

Conservation laws protect dynamic spin correlations from decay: Limited role of integrability in the central spin model

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Mazur's inequality renders statements about persistent correlations possible. We generalize it in a convenient form applicable to any set of linearly independent constants of motion. This approach is used to show rigorously that a fraction of the initial spin correlations persists indefinitely in the isotropic central spin model unless the average coupling vanishes. The central spin model describes a major mechanism of decoherence in a large class of potential realizations of quantum bits. Thus the derived results contribute significantly to the understanding of the preservation of coherence. We will show that persisting quantum correlations are not linked to the integrability of the model, but caused by a finite operator overlap with a finite set of constants of motion.

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a. Introduction. The two-time correlation function of two observables reveals important information about the dynamics of a system in and out of equilibrium: The noise spectra are obtained from symmetric combinations of correlation functions, while the causal, antisymmetric combination determines the susceptibilities required for the theory of linear response.

The two-time correlation function only depends on the time difference if at $t = 0$ the system of interest is prepared in a stationary state whose density operator commutes with the time-independent Hamiltonian. This is what will be considered in this work. Since correlations generically decay for $t \rightarrow \infty$, important information about the system dynamics is gained if a non-decaying fraction of correlations prevails at infinite times. Such non-decaying correlations are clearly connected to a limited dynamics in certain subspaces of the Hilbert space. The question arises if such a restricted dynamics is always linked to the integrability of the Hamiltonian. Here integrability means that the Hamiltonian can be diagonalized by Bethe ansatz which implies that there is an extensive number of constants of motion. Identifying and understanding those non-decaying correlations can be potentially exploited in applications for persistent storage of (quantum) information.

In this Letter we first prove that persisting correlations are not restricted to integrable systems by using a generalized form of Mazur's inequality^{1,2}. This is in contrast to the behavior of the Drude weight in the frequency-dependent conductivity of one-dimensional systems which appears to vanish abruptly once the integrability is lost, even if only by including an arbitrarily small perturbation. So far, the Drude weight has been the most common application of Mazur's inequality, see for instance Refs. 3–6 and references therein. Second, we apply this approach to the central spin model (CSM)⁷ describing the interaction of a single spin, e.g., an electronic spin in a quantum dot^{8,9}, an effective two-level

model in a NV center in diamond¹⁰, or a ¹³C nuclear spin¹¹, coupled to a bath of surrounding nuclear spins inducing decoherence.

Persisting spin correlations have been found in the CSM by averaging the central spin dynamics over a bath of random classical spins^{12,13} or in Markov approximation^{14,15}. Finite-size calculations^{16,17} of the full quantum problem and stochastic evaluation¹⁸ of the exact Bethe ansatz equations⁷ for small system sizes ($N \leq 48$) have also provided evidence for a non-decaying fraction of the central spin correlation, predicting a non-universal, system dependent value. Its origin has remained obscure, and it has been speculated that the lack of spin decay might be linked to Bose-Einstein condensate-like physics¹⁸.

While it is fascinating to identify such non-decaying correlations, it is technically very difficult to rigorously establish them. Approximate methods often miss precisely those intricate aspects allowing correlations to persist, especially when they explicitly exploit the assumption that the system relaxes towards a statistical mixture. Numerical approaches are either restricted in system size^{17,19}, or they are limited in the maximum time which can reliably be captured^{16,17,20}. Even analytical solutions⁷ can often only be evaluated in small systems¹⁸. Thus, a rigorous result establishing the existence of non-decaying correlations is highly desirable and we resort to Mazur's inequality for this purpose.

b. General Derivation. To establish the key idea and to fix the notation we present the following modified derivation related to Suzuki's derivation in Ref. 2. We consider the time-independent Hamiltonian H and the operator A with a vanishing expectation value $\langle A \rangle = 0$ with respect to a stationary density operator ρ , i.e. $[\rho, H] = 0$ so that two-time correlation functions only depend on the time difference. Note that ρ does not need to be the equilibrium density operator. Then, ρ and H have a complete common eigenbasis $\{|j\rangle\}$ in a finite-

dimensional Hilbert space, and their spectra are $\{\rho_j > 0\}$ and $\{E_j\}$, respectively. We define the correlation function of A as

$$S(t) := \langle A^\dagger(t)A(0) \rangle = \text{Tr} [\rho A^\dagger(t)A(0)] \quad (1a)$$

$$= \sum_{j,m} \rho_j |A_{jm}|^2 \exp(i(E_j - E_m)t), \quad (1b)$$

so that Eq. (1b) is its Lehmann representation, and $A_{jm} := \langle j|A|m \rangle$ denotes the matrix element of A . Physically, $S(t)$ stands for a measurement of A^\dagger at time t after the evolution from the initial state prepared by applying A at $t = 0$. Especially, for $A = S^z$ of a spin $S = 1/2$ in a disordered environment, $S(t)$ is proportional to $\langle S^z(t) \rangle$ if $\langle S^z(0) \rangle = 1/2$, see Supplement A for details. If $\lim_{t \rightarrow \infty} S(t)$ exists, it is given by

$$S_\infty := \sum_{jm} \rho_j |A_{jm}|^2 \delta_{E_j, E_m} \geq 0. \quad (2)$$

If $S(t \rightarrow \infty)$ does not exist, and $|S(t)| < \infty$, the long-time average $\lim_{T \rightarrow \infty} T^{-1} \int_0^T S(t) dt = S_\infty$ is projecting out the time-independent part S_∞ and uniquely defines the non-decaying fraction of the correlation.

In practice, the Lehmann representation (1b) requires the complete diagonalization of H which is not feasible for large systems. Hence one resorts to constants of motion. To this end, we define the scalar product for two operators X and Y as

$$\langle X|Y \rangle := \langle X^\dagger Y \rangle = \text{Tr} [\rho X^\dagger Y] \quad (3)$$

in the super-Hilbert space of the operators. If a set of M conserved linearly independent operators X_i with $[X_i, H] = 0$ is known, one may assume their orthonormality $\langle X_i|X_m \rangle = \delta_{im}$ provided by a Gram-Schmidt process. Then, we expand the operator of interest A

$$A = \sum_{i=1}^M a_i X_i + R \quad (4)$$

in this incomplete operator basis where $a_i := \langle X_i|A \rangle$ and R is the remaining rest with $\langle X_i|R \rangle = 0 \ \forall i \in \{1, \dots, M\}$. Substituting (4) into the definition (1a) yields

$$S(t) = \sum_{i=1}^M |a_i|^2 + S^{(R)}(t) \quad (5)$$

with $S^{(R)}(t) := \langle R(t)R(0) \rangle$. This relies on the constancy of (i) $\langle X_i^\dagger(t)X_m(0) \rangle = \delta_{im}$, of (ii) $\langle X_i^\dagger(t)R(0) \rangle = 0$, and of (iii) $\langle R^\dagger(t)X_m(0) \rangle = 0$ all stemming from $[X_j, H] = 0$. For the last relation we have used the cyclic invariance of the trace and $[\rho, H] = 0$.

If we knew $\lim_{t \rightarrow \infty} S^{(R)}(t) = 0$, we would deduce $S_\infty = \sum_{i=1}^M |a_i|^2$. But in general this does not hold because R may still contain a non-decaying part. But (5) implies Mazur's inequality

$$S_\infty \geq S_{\text{low}} := \sum_{i=1}^M |a_i|^2. \quad (6)$$

For a given H , the complete set of conserved operators Γ is spanned by all pairs of energy-degenerate eigenstates

$$\Gamma := \{|j\rangle\langle m|/\sqrt{\rho_m} \text{ with } E_j = E_m\}. \quad (7)$$

The elements of Γ are orthonormal with respect to the scalar product (3). The coefficient a_{jm} of $X_{jm} = |j\rangle\langle m|/\sqrt{\rho_m}$ takes the value $\sqrt{\rho_m}A_{jm}$ so that the right hand side of (6) equals S_∞ as given by the Lehmann representation (2). Thus, the inequality (6) is tight because it becomes exact for the *complete* set Γ of conserved operators. The physical interpretation of Eq. (6) is straightforward in the Heisenberg picture if we view the time-dependent observable A^\dagger as super vector. Its components parallel to conserved quantities (super vector directions) are constant in time because these quantities commute with the Hamiltonian. But all other components, which are perpendicular to the conserved super subspace, finally decay.

If not all conserved operators are considered, the r.h.s. of (6) decreases and only the inequality holds. Generally, if *any* subspace of the space spanned by Γ is considered Mazur's inequality (6) holds. One does not need to know the complete set of eigenstates of H in order to calculate a lower bound: Any finite (sub)set of conserved operators is sufficient.

Now we proceed to generalize Mazur's inequality for easy-to-use application. Usually, some conserved operators C_i are known but they are not necessarily orthonormal in general. Rather their overlaps yield a Hermitian, positive norm matrix \mathbf{N} with matrix elements $N_{im} := \langle C_i|C_m \rangle$. Each operator C_i can be represented as a linear superposition of the complete set of orthonormal X_i . These superpositions can be summarized in a matrix \mathbf{M} so that $\mathbf{c} = \mathbf{M}^* \mathbf{x}$ where the vectors \mathbf{x} and \mathbf{c} contain the operators X_i and C_i as coefficients; \mathbf{M}^* is the complex (not Hermitian!) conjugate of \mathbf{M} . A short calculation shows that $\mathbf{N} = \mathbf{M} \mathbf{M}^\dagger$.

If we define the vector \mathbf{a}_X with complex components a_i , the bound S_{low} can be expressed by $S_{\text{low}} = \mathbf{a}_X^\dagger \mathbf{a}_X$. In analogy, we compute \mathbf{a}_C with complex components $\langle C_i|A \rangle$. Obviously, $\mathbf{a}_X = \mathbf{M}^{-1} \mathbf{a}_C$ holds and the lower bound is computed by

$$S_{\text{low}} = \mathbf{a}_C^\dagger (\mathbf{M}^{-1})^\dagger \mathbf{M}^{-1} \mathbf{a}_C = \mathbf{a}_C^\dagger \mathbf{N}^{-1} \mathbf{a}_C \quad (8)$$

without resorting to orthonormalized operators, relying only on the scalar products of C_i and A . We have successfully eliminated the construction of a subset of orthogonal operators X_i and related the lower bound to some known set of linear independent unnormalized conserved operators C_i . The general lower bound (8) is our first key result. A possible route to generalizations to various initial states is sketched in the Supplement.

c. Central spin model. The Hamiltonian of the CSM reads

$$H_0 = \vec{S}_0 \cdot \sum_{k=1}^N J_k \vec{S}_k \quad (9)$$

where we assume all spins to be $S = 1/2$ for simplicity. It is a generic model to study the interaction between a two-level system and a bath of spins or more generally a set of subsystems with finite number of levels. Currently, it is intensively investigated for understanding the decoherence and dephasing in possible realizations of quantum bits^{8,9,22,23}. Theoretical tools comprise Chebyshev polynomial technique^{17,24}, perturbative approaches^{15,25,26}, generalized Master equations^{27–29}, equations of motion³⁰ various cluster expansions^{31–34}, Bethe ansatz^{7,18,35,36}, density-matrix renormalization¹⁶, and studies of the classical analogue^{12,13,37,38}.

By focusing on $A = S_0^z$, the correlation function defined in (1a) reveals important information on the decay of the central spin. Due to isotropy no other components of the central spin need to be considered. Given the smallness of the hyperfine couplings (J_k is in the range of μeV corresponding to percents of a Kelvin^{8,9,12,22,23}) the experimentally relevant temperature can be considered as infinite, and we take the spin system to be completely disordered, i.e., $\rho \propto \mathbb{1}$, prior to the preparation of an initial state of the central spin, cf. Supplement.

For classical spins S_k , there are strong analytical arguments that a fraction of central spin correlations persists unless there is a diverging number of arbitrarily weakly coupled spins in the bath^{12,13,37}. In the quantum case smaller systems have been studied and evidence for a non-decaying fraction of spin polarization^{17,18} has only been compiled in fairly small ($N < 50$) systems or up to fairly short times¹⁶.

Based on the generalized Mazur's inequality (8), we are able to address the nature and the lower bound of these non-decaying correlations for arbitrary system sizes. The total spin $\vec{I} := \sum_{k=0}^N \vec{S}_k$ could serve as a first guess for a useful conserved quantity. Only the z -component $C_1 := I^z$ has an overlap $\mathbf{a} = (I^z | S_0^z) = 1/4$ (we omit the subscript C for brevity). The norm $N_{11} = (I^z | I^z)$ takes the value $(N+1)/4$ so that (8) provides $S_{\text{low}} = 1/(4(N+1))$. Irrespective of the considered distribution of the couplings J_k , using only I^z as single conserved operator does not provide a meaningful lower bound for thermodynamically large, or infinite baths.

The next important conserved quantity is the energy H_0 itself. But, of course, $(H_0 | S_0^z) = 0$ because H_0 is a scalar and S_0^z a vector component. The z -component of the product $\vec{I}H_0$, $H_0^z := I^z H_0$, clearly fulfills $[H_0, H_0^z] = 0$ and defines a conserved composite vector operator. We find

$$(S_0^z | H_0^z) = J_S/16 \quad (10a)$$

$$(H_0^z | H_0^z) = (2J_S^2 + 3(N-1)J_Q^2)/64 \quad (10b)$$

where $J_S := \sum_{k=1}^N J_k$ and $J_Q^2 := \sum_{k=1}^N J_k^2$. With this input Eq. (8) yields

$$S_{\text{low}} = \frac{1}{4} \frac{J_S^2}{2J_S^2 + 3(N-1)J_Q^2}. \quad (11)$$

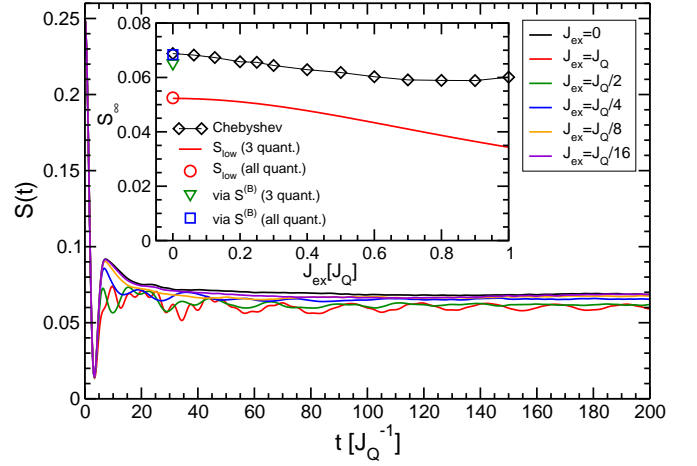


FIG. 1. (Color online) Spin correlation $S(t)$ for $N = 20$ bath spins with $J_k \propto k$, but normalized such that $J_Q = 1$ is the unit of energy, and various J_{ex} defined in (12). The inset compares S_∞ from the average of the numerical data with $t \in [150/J_Q, 200/J_Q]$ to S_{low} obtained from (8) for the 3 quantities (I^z, I_Q^z, H_0^z) or for all quantities (I^z, H_l^z with $l \in \{1, 2, \dots, N\}$). The estimates from the Overhauser correlations $S^{(B)}$ are also shown.

This bound remains finite for $N \rightarrow \infty$ if the J_k are drawn from a probability distribution $p(J)$ with average \bar{J} and variance $\overline{\Delta J^2}$. For large N one has $J_S = N\bar{J}$ and $J_Q^2 = N(\bar{J}^2 + \overline{\Delta J^2})$ so that $S_{\text{low}} = \bar{J}^2/[20\bar{J}^2 + 12\overline{\Delta J^2}]$ ensues for $N \rightarrow \infty$. This is a finite lower bound unless the average values \bar{J} vanishes. This rigorous bound is our second key result.

For any finite system with non-vanishing sum J_S , Eq. (11) provides a rigorous finite lower bound which is very easy to compute for any given set of couplings. It can serve to check the validity of numerical results such as provided in Refs.^{17,18}. Generically, distributions of the J_k have finite values \bar{J} and $\overline{\Delta J^2}$. This is the case for nuclear spins in molecules¹¹ or NV centers in diamond¹⁰ because the spin baths are finite. In quantum dots, the convergence and existence J_S and J_Q is ensured even for arbitrary number of spins because the couplings are bounded from above, but become arbitrarily small due to exponential tails of the electron wave function^{8,9,12,22,23}. This leads to vanishing \bar{J} implying complete decay for infinite times.

For large, but finite times, however, our results include the possibility of slow decays $S(t) \propto \ln(t)^{-\alpha}$ previously advocated for infinitely large spin baths^{13,37,39}. Assuming exponential scaling for the couplings $J_k \propto \exp(-\beta k)$, where β is inversely proportional to the number of relevant bath spins⁴⁰, it is clear that J_S and J_Q^2 converge quickly for $N \rightarrow \infty$ so that Eq. (11) implies $S_{\text{low}} \propto 1/N$. Chen et al.¹³ have argued that at any given *finite* time t , only those spins \vec{S}_k with couplings $tJ_k \gtrsim 1$ significantly influence the real-time dynamics of the central

spin. Hence, only an effective number $N_{\text{eff}}(t) \propto \ln(t)$ of spins contribute to the correlation function implying $S(t) \propto 1/\ln(t)$ for such a distribution function.

The lower bound (11) can be improved by considering the three conserved observables I^z , H_0^z , and $I_Q^z := I^z \sum_{i < j} \vec{S}_i \cdot \vec{S}_j$. The required vector and matrix elements are given in the supplemental material. Still the bound does not exhaust the numerically found value as depicted in the inset of Fig. 1 for $J_{\text{ex}} = 0$ (J_{ex} makes the system non-integrable, it will be defined in (12)). Even resorting to the integrability of the CSM⁷ which implies $0 = [H_l, H_p]$ with $H_l := \sum_{k=0, \neq l}^N (\varepsilon_l - \varepsilon_k)^{-1} \vec{S}_l \cdot \vec{S}_k$ and $\varepsilon_0 = 0, \varepsilon_k = -1/J_k$ does not account for the full non-decaying fraction obtained in finite size calculations¹⁷, see circle in the inset of Fig. 1. The bound has been computed considering I^z and $H_l^z := I^z H_l$ for $l \in \{1, 2, \dots, N\}$ (for matrix elements see supplement).

The above results suggest that the integrability is not the key ingredient for a finite non-decaying fraction. To support this claim we extend the Hamiltonian (9) by adding one extra coupling $H_0 \rightarrow H$

$$H := H_0 + J_{\text{ex}} \vec{S}_1 \cdot \vec{S}_N \quad (12)$$

between the most weakly and the most strongly coupled bath spin, defined to be at $k = 1$ and N , respectively. Its value J_{ex} is chosen to be $\mathcal{O}(J_Q)$ so that it constitutes a sizable perturbation even for large spin baths.

The modified time-dependence of $S(t)$ is depicted for various J_{ex} in Fig. 1. A finite J_{ex} spoils the integrability completely⁷, but leaves the quantities I^z, I_Q^z, H^z conserved. These three constants of motion generic for isotropic spin models are used to obtain the lower bound (red curve) in the inset of Fig. 1. Obviously, S_{low} is decreased smoothly and only moderately upon increasing J_{ex} in line with the numerically determined S_{∞} . There is no abrupt jump to zero, in contrast to what is known for the Drude weight. The conclusion that integrability is only secondary for the non-decaying spin correlation is our third key result.

At present it remains an open question which conserved quantities one has to include to yield a tight lower bound. We presume that higher powers of H , for instance $I^z H^2$, have to be considered. Such studies are more tedious and left for future research. Instead, we take a mathematically less rigorous route based on the estimate by Merkulov et al.¹²

$$S_{\infty} = S_{\infty}^{(B)} / (12S^{(B)}(0)) \quad (13)$$

where $S^{(B)}(t)$ is the correlation of the Overhauser field operator $\vec{B}_N := \sum_{j=0}^N J_k \vec{S}_k$. Note that an arbitrary J_0 can be included because $\vec{S}_0 \cdot \vec{B}_N$ differs from H_0 in (9) only by an irrelevant constant for spin 1/2. This estimate was derived for a classical, large Overhauser field¹² and prevails in the thermodynamic limit of the quantum case: The Overhauser field becomes a classical variable upon $N \rightarrow \infty$ as shown in Ref.¹⁶.

Thus we now apply the general approach (8) to $A = B_N^z$. Considering only $C_1 = I^z$ as conserved operator already yields a meaningful lower bound for the Overhauser field correlation function for $N \rightarrow \infty$

$$\frac{S_{\text{low}}^{(B)}}{S^{(B)}(0)} = \frac{(J_S + J_0)^2}{(N+1)(J_Q^2 + J_0^2)}. \quad (14)$$

Recall $J_S \propto N$ and $J_Q^2 \propto N$ if the couplings are drawn from a normalized distribution function $p(J)$. This lower bound can be optimized by choosing the arbitrary value J_0 such that the bound becomes maximal. With the matrix elements given in the supplement $S_{\text{low}}^{(B)}$ can be improved considering the three constants I^z, I_Q^z, H^z or all integrals I^z and $H_l^z, 1 \leq l \leq N$. The results are also included in Fig. 1 (triangle and square symbols). They hold only for $J_{\text{ex}} = 0$ because the estimate (13) applies only in this case. Remarkably, the resulting estimates for S_{∞} seem to be tight. In particular, the easily evaluated estimate based on all integrals reproduces the numerically found S_{∞} to its accuracy. We applied the same estimate to the case $J_k \propto \exp(-\beta k)$ studied by stochastically evaluating the Bethe ansatz equations and found excellent agreement with the published data with $N \leq 48$ in Ref.¹⁸ as well. Thus we conjecture that the non-decaying fraction S_{∞} in the central spin model is quantitatively described by $S_{\text{low}}^{(B)} / (12S^{(B)}(0))$ if $S_{\text{low}}^{(B)}$ is determined from the $N+1$ integrals I^z and H_l^z . This constitutes our fourth key result. The small difference, however, between triangle (from three constants of motion) and square (from $N+1$ constants of motion) in Fig. 1 indicates again that the significance of the integrability is limited.

In summary, four key results are obtained: (i) An easy-to-use version of Mazur's inequality to prove persisting correlations; (ii) A rigorous finite lower bound for the infinite-time spin correlation in the CSM, valid for the infinite system if the average coupling is finite; (iii) Only a small part of the persisting correlation is due to the integrability; (iv) A quantitative estimate for the persisting correlation is conjectured, based on the Overhauser field.

Clearly, the generalized inequality calls for application to other problems⁴¹. The approach is easy to evaluate and can be used for very large systems and large numbers of constants of motion. Thus it can prove fruitful in the intensely studied field of integrable systems, for instance in estimating Drude weights. In the context of coherence in particular, various extensions of the CSM, e.g., by magnetic fields, anisotropies, or more intra-bath couplings suggest themselves to be investigated in the presented manner.

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I. SUPPLEMENTAL MATERIAL

A. Time-Dependent Expectation Values

One may wonder whether the two-time correlations $S(t) = \langle A^\dagger(t)A(0) \rangle$ reflect time-dependent measurements after the preparation of some initial state. We show that this is the case for the simple, but important example of a spin correlation for $S = 1/2$, i.e., for $A = S_0^z$. Then we write $S_0^z = \frac{1}{2}(P_+ - P_-)$ where P_σ projects onto the states with $S_0^z = \sigma/2$. If ρ denotes the density matrix of the total system before any state preparation we calculate

$$S(t) = \langle S_0^z(t)S_0^z(0) \rangle \quad (15a)$$

$$= \frac{1}{2} \langle S_0^z(t)(P_+ - P_-) \rangle \quad (15b)$$

$$= \frac{1}{2} \langle S_0^z(t)P_+ \rangle - \frac{1}{2} \langle S_0^z(t)P_- \rangle \quad (15c)$$

$$= \langle S_0^z(t)P_+ \rangle \quad (15d)$$

$$= \text{Tr}(S_0^z(t)P_+\rho) \quad (15e)$$

$$= \frac{1}{2} \text{Tr}(S_0^z(t)\rho_{\text{initial}}) \quad (15f)$$

$$= \frac{1}{2} \langle S_0^z(t) \rangle_{\text{initial}} \quad (15g)$$

where we assumed that the Hamiltonian H and the density matrix ρ are invariant under total inversion $S^z \rightarrow -S^z$ so that the second term in (15c) equals the first

one. Finally, in (15f) we define the initial density matrix $\rho_{\text{initial}} := (1/2)P_{+\rho}$ which results from ρ by projecting it to the states with $S_0^z = 1/2$ and its proper normalization. This clearly shows that in the studied case $S(t)$ equals the time-dependent expectation value for a suitably prepared initial state.

The above procedure can be modified to other observables. Generally, we can consider $\langle A^\dagger(t)D(0) \rangle$ to focus on the time-dependent expectation value $\langle A^\dagger(t) \rangle_D$ starting from the initial density matrix $\rho_{\text{initial}} := D\rho$. However, do not claim that a suitable operator D is easy to find. This route remains to be explored in future work.

B. Rigorous Bound for Non-Decaying Spin Correlation

For completeness, we recall the following definitions of conserved quantities. The total angular momentum \vec{I} and the combination \vec{I}_Q derived from it read

$$\vec{I} := \sum_{j=0}^N \vec{S}_j \quad (16a)$$

$$\vec{I}_Q := \sum_{j=0}^N \vec{S}_j \sum_{0 \leq l < p \leq N} (\vec{S}_l \cdot \vec{S}_p). \quad (16b)$$

Below we only need the corresponding z -components. Furthermore, we consider

$$H_l^z := \sum_{j=0}^N S_j^z \sum_{j=0, \neq l}^N J_j^{(l)} (\vec{S}_l \cdot \vec{S}_j) \quad (16c)$$

based on the constants of motion $H_l = \sum_{j=0, \neq l}^N J_j^{(l)} (\vec{S}_l \cdot \vec{S}_j)$ of the integrable CSM⁷ where we use the shorthand $J_j^{(l)}$ and introduce some further shorthands for future use

$$J_j^{(l)} := (\varepsilon_l - \varepsilon_j)^{-1} \quad (17a)$$

$$S^{(l)} := \sum_{j=0, \neq l}^N J_j^{(l)} \quad (17b)$$

$$Q^{(l)} := \sum_{j=0, \neq l}^N \left(J_j^{(l)} \right)^2 \quad (17c)$$

where $\varepsilon_0 = 0$ and $\varepsilon_j = -J_j^{-1}$. Note that $J_j = J_j^{(0)}$, $J_S = S^{(0)}$, and $J_Q^2 = Q^{(0)}$.

For the disordered spin system with density operator ρ proportional to the identity the following diagonal scalar products can be determined straightforwardly

$$\langle I^z | I^z \rangle = (N+1)/4 \quad (18a)$$

$$\langle I_Q^z | I_Q^z \rangle = (N+1)N(7N-5)/128 \quad (18b)$$

$$\langle H_l^z | H_l^z \rangle = (2(S^{(l)})^2 + 3(N-1)Q^{(l)})/64. \quad (18c)$$

We also need the non-diagonal matrix elements

$$\langle I^z | I_Q^z \rangle = (N+1)N/16 \quad (19a)$$

$$\langle I^z | H_l^z \rangle = S^{(l)}/8 \quad (19b)$$

$$\langle I_Q^z | H_0^z \rangle = J_S(7N-5)/64 \quad (19c)$$

$$\begin{aligned} \langle H_l^z | H_p^z \rangle &= J_p^{(l)}(S^{(p)} - S^{(l)})/16 \\ &\quad - 3(N-3)(J_p^{(l)})^2/64 \quad \text{for } l \neq p. \end{aligned} \quad (19d)$$

For the observable S_0^z we obtain the vector elements

$$\langle S_0^z | I^z \rangle = 1/4 \quad (20a)$$

$$\langle S_0^z | I_Q^z \rangle = N/16 \quad (20b)$$

$$\langle S_0^z | H_0^z \rangle = J_S/16 \quad (20c)$$

$$\langle S_0^z | H_l^z \rangle = -J_l/16 \quad \text{for } l > 0. \quad (20d)$$

With these matrix and vector elements we can compute S_{low} in (8) for various sets of conserved quantities. Note that H_0^z is linearly dependent on the N quantities H_l^z with $0 < l \leq N$ due to

$$\sum_{l=0}^N H_l^z = 0. \quad (21)$$

Similarly, I_Q^z depends linearly on them due to

$$I_Q^z = \sum_{l=1}^N \varepsilon_l H_l^z. \quad (22)$$

Hence, one may either consider I^z together with the N quantities H_l^z with $0 < l \leq N$ or the three quantities I^z, I_Q^z, H_0^z . The first choice exploits all the known conserved quantities on the considered level of at most trilinear spin combinations. This is what is called ‘all quantities’ in Fig. 1 in the Letter. No explicit formula can be given, but the required matrix inversion is easily performed for up to $N = O(1000)$ spins with any computer algebra program and up to $N \approx 10^6$ spins by any subroutine package for linear algebra.

The second choice of I^z, I_Q^z, H_0^z yields 3×3 matrices and can be analysed analytically. Inserting the elements in (18) and in (19) and those in (20) into (8) yields

$$S_{\text{low}} = \frac{1}{4(N+1)} \frac{(3J_Q^2 + J_S^2)N(N+1) - 10J_S^2}{3J_Q^2N(N+1) + 2J_S^2(N-5)}. \quad (23)$$

Furthermore, these three quantities are conserved for any isotropic spin model so that we may also consider the system with the additional bond $H = H_0 + J_{\text{ex}}\vec{S}_1 \cdot \vec{S}_N$, see Fig. 1. Thus we extend the above formulae by passing from H_0 to H and hence from H_0^z to $H^z = I^z H$. The modified scalar products are

$$\begin{aligned} \langle H^z | H^z \rangle &= \langle H_0^z | H_0^z \rangle \\ &\quad + (J_1 + J_N)J_{\text{ex}}/16 + (3N-1)J_{\text{ex}}^2/64 \end{aligned} \quad (24a)$$

$$\langle I^z | H^z \rangle = \langle I^z | H_0^z \rangle + J_{\text{ex}}/8 \quad (24b)$$

$$\langle I_Q^z | H^z \rangle = \langle I_Q^z | H_0^z \rangle + (7N-5)J_{\text{ex}}/64 \quad (24c)$$

$$\langle S_0^z | H^z \rangle = \langle S_0^z | H_0^z \rangle. \quad (24d)$$

They lead to a bound $S_{\text{low}}(J_{\text{ex}})$ as depicted in Fig. 1. The explicit formula is similar to the one in (23), but lengthy so that we do not present it here. It can be easily computed by computer algebra programs.

C. Estimate via Bound for the Overhauser Field

Eq. (13) relates the non-decaying fraction S_{∞} to the relative bound for the Overhauser field

$$\vec{B} = \sum_{j=0}^N J_j \vec{S}_j \quad (25)$$

where J_0 is arbitrary if the central spin has $S = 1/2$. We stress, however, that the derivation yielding (13) in Ref. 12 only holds for the CSM so that we do not consider extensions to finite J_{ex} in this case.

We use the freedom to choose J_0 to maximize the resulting lower bound for $A = B^z$. We reuse all matrix elements of the norm matrix \mathbf{N} in (18) and in (19). Since (13) uses the relative correlation we have to compute

$$S^{(B)}(t=0) = (B^z|B^z) = (J_Q^2 + J_0^2)/4 \quad (26)$$

as well. Furthermore, the vector elements of \mathbf{a} must be

determined anew

$$(B^z|I^z) = (J_S + J_0)/4 \quad (27a)$$

$$(B^z|I_Q^z) = (J_S + J_0)N/16 \quad (27b)$$

$$(B^z|H_0^z) = (J_Q^2 + J_0 J_S)/16 \quad (27c)$$

$$(B^z|H_l^z) = J_l(S^{(l)} + J_l)/8 - J_l(J_S + J_0)/16 \quad \text{for } l > 0. \quad (27d)$$

These elements allow us to determine the ratio $S_{\text{low}}^{(B)}/S^{(B)}(0)$ for the three quantities I^z, I_Q^z, H_0^z or for all quantities, i.e., I^z and H_l^z with $1 \leq l \leq N$. The ensuing lower bounds can be optimized by varying J_0 in such a way that the ratios become maximum yielding the best bounds. The latter step is easy to perform since the non-linear equation in J_0 to be solved to determine the maximum is just a quadratic one. In this way, the triangle and square symbols in Fig. 1 are computed.

The comparison to the Bethe ansatz data for up to $N = 48$ spins in Ref.¹⁸ yields an excellent agreement within the accuracy with which we can read off S_{∞} from the numerically evaluated Bethe ansatz correlation $S(t)$. This concludes the section on the required input of matrix and vector elements.